Cardinal of Continuum

Cantor's most brilliant insight is that some infinities are bigger than others. He discovered something remarkable about the 'continuum'—a fancy name for the real number system. Its cardinal, which he denoted by $\mathfrak{c}$, is bigger than $\aleph_0$. I don't just mean that some real numbers aren't whole numbers. Some rational numbers (in fact, most) aren't whole numbers, but the integers and the rationals have the *same* cardinal, $\aleph_0$. For infinite cardinals, the whole need not be greater than the part, as Galileo realized. It means that you can't match all the real numbers one to one with all the whole numbers, no matter how you jumble them up.

Since $\mathfrak{c}$ is bigger than $\aleph_0$, Cantor wondered if there were any infinite cardinals in between. His continuum hypothesis states that there aren't. He could neither prove nor disprove this contention. Between them, Kurt Gödel in 1940 and Paul Cohen in 1963 proved that the answer is 'yes and no'. It depends on how you set up the logical foundations of mathematics.

Uncountable Infinity

Recall that a real number can be written as a decimal, which can either stop after finitely many digits, like 1·44, or go on forever, like π. Cantor realized (though not in these terms) that the infinity of real numbers is definitely larger than that of the counting numbers, $\aleph_0$.

The idea is deceptively simple. It uses proof by contradiction. Suppose, in the hope of deriving a logical contradiction, that the real numbers can be matched to the counting numbers. Then there is a list of infinite decimals, of the form

$1 \leftrightarrow a_0 . \mathbf{a_1}\ a_2\ a_3\ a_4\ a_5 \ldots$
$2 \leftrightarrow b_0 . b_1\ \mathbf{b_2}\ b_3\ b_4\ b_5 \ldots$
$3 \leftrightarrow c_0 . c_1\ c_2\ \mathbf{c_3}\ c_4\ c_5 \ldots$
$4 \leftrightarrow d_0 . d_1\ d_2\ d_3\ \mathbf{d_4}\ d_5 \ldots$
$5 \leftrightarrow e_0 . e_1\ e_2\ e_3\ e_4\ \mathbf{e_5} \ldots$

such that every possible infinite decimal appears somewhere on the right-hand side. Ignore the boldface for a moment; I'll come to that soon.

Cantor's bright idea is to construct an infinite decimal that cannot possibly appear. It takes the form

$0 . x_1\ x_2\ x_3\ x_4\ x_5 \ldots$

where

x_1 is different from a_1

x_2 is different from b_2

x_3 is different from c_3

x_4 is different from d_4

x_5 is different from e_5

and so on. These are the digits I marked in boldface type.

The main point here is that if you take an infinite decimal and change just one of its digits, however far along, you change its value. Not by much, perhaps, but that's not important. What matters is that it's changed. We get our new 'missing' number by playing this game with every number on the allegedly complete list.

The condition on x_1 means that this new number is not the first in the list, because it has the wrong digit in the first place after the decimal point. The condition on x_2 means that this new number is not the second in the list, because it has the wrong digit in the second place after the decimal point. And so on. Because both the decimals and the list continue indefinitely, the conclusion is that the new number is *nowhere* on the list.

But our assumption was: it is on the list. This is a contradiction, so our assumption is wrong, and no such list exists.

One technical issue needs attention: avoid using either 0 or 9 as digits in the number under construction, because decimal notation is ambiguous. For example, 0.10000… is exactly the same number as 0.09999… (they are two distinct ways to write $\frac{1}{10}$ as an infinite decimal). This ambiguity occurs only when the decimal ends in an infinite sequence of 0s or an infinite sequence of 9s.

This idea is called Cantor's diagonal argument, because the digits a_1, b_2, c_3, d_4, e_5, and so on run along the diagonal of the right-hand side of the list. (Look at where the boldface digits occur.) The proof works precisely because both the digits, and the list, can be matched to the counting numbers.

It's important to understand the logic of this proof. Admittedly, we can deal with the particular number that we constructed by sticking it on the top of the list and moving all the others down one space. But the logic of proof by contradiction is that we've already assumed that won't be necessary. The number we construct is supposed to be in the list already, without further modification. But it's not. Therefore: no such list.

Since every whole number is a real number, this implies that in Cantor's set-up the infinity of all real numbers is bigger than the infinity of all whole numbers. By modifying the Russell paradox, he went much further, proving that there is no largest infinite number. That led him to envisage an infinite series of ever-larger infinite numbers, known as infinite (or transfinite) *cardinals*.

No Largest Infinity

Cantor thought that his series of infinite numbers ought to start out like this:

$\aleph_0$ $\aleph_1$ $\aleph_2$ $\aleph_3$ $\aleph_4$…

with each successive infinite number being the 'next' one, in the sense that there aren't any in between. The whole numbers correspond to $\aleph_0$. So do the rational numbers. But real numbers need not be rational. Cantor's diagonal argument proves that is bigger than $\aleph_0$, so presumably the real numbers should correspond to $\aleph_1$. But do they?

The proof doesn't tell us that. It says that $\mathfrak{c}$ is bigger than $\aleph_0$, but it doesn't rule out the possibility that something else lies in between them. For all Cantor knew, $\mathfrak{c}$ might be, say, $\aleph_3$. Or worse.

He could prove some of this. Infinite cardinals can indeed be arranged in that manner. Moreover, the subscripts 0, 1, 2, 3, 4, … don't stop with finite whole numbers. There must also be a transfinite number, for instance: it's the smallest transfinite number that's bigger than all of the n with n any whole number. And if things stopped there, it would violate his theorem that there is no largest transfinite number, so they don't stop. Ever.

What he couldn't prove was that the real numbers correspond to $\aleph_1$. Maybe they were $\aleph_2$ and some other set was in between, so that set was $\aleph_1$. Try as he might, he couldn't find such a set, but he couldn't prove it didn't exist. Where were the real numbers in his list of alephs? He had no idea. He suspected that the real numbers did indeed correspond to $\aleph_1$, but this was pure conjecture. So he ended up using a different symbol: gothic $\mathfrak{c}$, which stands for 'continuum', the name then used for the set of all real numbers.

A finite set with n elements has 2^n different subsets. So Cantor defined 2^A, for any cardinal A, by taking some set with cardinal A and defining 2^A to be the cardinal of the set of all subsets of that set. Then he could prove that 2^A is bigger than A for any infinite cardinal A. Which, incidentally, implies that there is no biggest infinite cardinal. He could also prove that $\mathfrak{c} = 2^{\aleph_0}$. It seemed likely that $\aleph_{n+1} = 2^{\aleph_n}$. That is, taking the set of all subsets leads to the next largest infinite cardinal. But he couldn't prove that.

He couldn't even prove the simplest case, when n = 0, which is equivalent to stating that $\mathfrak{c} = \aleph_1$. In 1878 Cantor conjectured that this equation is true, and it became known as the continuum hypothesis. In 1940 Gödel proved that the answer 'yes' is logically consistent with the usual assumptions of set theory, which was encouraging. But then, in 1963, Cohen proved that the answer 'no' is *also* logically consistent.

Oops.

This is not a logical contradiction in mathematics. It's meaning is much stranger, and in some ways more disturbing: the answer depends on which version of set theory you use. There's more than one way to set up logical foundations for mathematics, and while all of them agree on the basic material, they can disagree about more advanced concepts. As Walt Kelly's cartoon character Pogo was wont to say: 'We have met the enemy and he is us.' Our insistence on axiomatic logic is biting us in the ankle.

Today we know that many other properties of infinite cardinals also depend on which version of set theory you use. Moreover, these questions have close links to other properties of sets that do not involve cardinals explicitly. The area is a happy hunting ground for mathematical logicians, but on the whole the rest of mathematics seems to work fine whichever version of set theory you use.

Excerpt From: Ian Stewart, "Professor Stewart's Incredible Numbers."